RIGOROUS RESULTS IN TOPOLOGICAL σ-MODEL

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We give here a review of certain recent develoments in the mathematical formulation of the two-dimensional topological σ-model introduced several years ago by E. Witten (see [W1,W2]). Our review is very far from being complete and covers only a small portion of the existing bibliography.

First of all, recall that the partition function of the two-dimensional σ-model is given by the following Feynman integral:

$$\int_{\phi:\Sigma\longrightarrow V} \exp\left(-\int_\Sigma |\phi'|^2+\dots\right)\mathcal{D}\phi$$

Here Σ is a closed surface of genus g endowed with a Riemannian metric and V is a Riemannian manifold. Dots in the action denote fermionic and topological terms.

The classical theory is conformally invariant, i.e. it depends only on the conformal/complex class of metric on Σ. Physicists beleive that the quantum theory is defined and it is conformally invariant for some *special class* of metrics on V close to the class of Einstein metrics. Notice that the standard regularization scheme of the Feynman integral by a discretization of the surface Σ fails because of the conformal invariance of the action functional.

Suppose that we have a metric on V giving a conformal field theory. Then, as usual, we have an infinite-dimensional pre-Hilbert space $\mathcal{H}$ of fields with a base $\{\mathcal{O}_\alpha\}$. Correlation functions of primary fields in the conformal field theory

$$\langle\mathcal{O}_{\alpha_1}(p_1)\dots\mathcal{O}_{\alpha_n}(p_n)\rangle_g$$

can be considered as real-analytic functions (more precisely, sections of some line bundles) on the space $\mathcal{M}_{g,n}$ of equivalence classes of surfaces Σ with conformal structures and n pairwise distinct points $p_1,\dots,p_n$ on Σ. This space is a non-compact orbifold of dimension $6g-6+2n$ and it is called the moduli space of smooth complex curves with punctures. Locally it looks like $\mathbf{C}^{3g-3+n}/\Gamma$ where Γ is a finite group acting linearly in a complex vector space. The space $\mathcal{M}_{g,n}$ is defined for all $g,n\geq 0$ except four cases: $g=0$, $n\leq 2$ or $g=1$, $n=0$.

If we include the supersymmetry into the action functional then we get correlators $\langle\mathcal{O}_{\alpha_1}\dots\mathcal{O}_{\alpha_n}\rangle_g$ for certain fields $\{\mathcal{O}_{\alpha_i}\}$ which are not functions on $\mathcal{M}_{g,n}$ but *differential forms* on it. BRST-closed fields give closed forms, and the space $\mathcal{H}^{BRST}$ of BRST-cohomology classes give cohomology classes on $\mathcal{M}_{g,n}$. Physical arguments show that these closed differential forms have smooth enough prolongation to the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. By definition, this compactification (intoduced in [DM]) is the moduli space of curves with marked points $(C; p_1, \dots, p_n)$ such that

(1) C is a compact connected complex curve,
(2) singularities of C are only simple self-crossing points,
(3) the Euler characteristic of the smooth part of C is equal to $2-2g$,
(4) marked points p_i are pairwise distinct and non-singular,
(5) the automorphism group of $(C; p_1, \dots, p_n)$ is finite.

Such curves are called stable marked curves.

The last condition in the definition of stable curves can be reformulated as

(1) there exists a complete hyperbolic metric on $C^{smooth} \setminus \{p_1, \dots, p_n\}$, or, equivalently,
(2) genus zero components of $C^{smooth} \setminus \{p_1, \dots, p_n\}$ have at least 3 punctures and genus 1 components have at least 1 puncture.

In string theory one wants to compute the following string correlation function:

$$\langle \mathcal{O}_{\alpha_1} \dots O_{\alpha_n} \rangle := \sum_{g \geq 0} \lambda^{2g-2} \int_{\overline{\mathcal{M}}_{g,n}} \langle \mathcal{O}_{\alpha_1} \dots O_{\alpha_n} \rangle_g \ ,$$

where λ is the string coupling constant.

In the topological sector of super-symmetric conformal string theory we integrate the top-degree component of closed differential forms over the fundamental class of $\overline{\mathcal{M}}_{g,n}$. Mathematically the arising structure was formalized in [KM]. Here is the list of axioms.

Axioms of topological σ-model.

DATA:

(1) $\mathcal{H}$: a finite-dimensional $\mathbf{Z}_2$-graded (=super) vector space over $\mathbf{C}$,
(2) $(\ ,\ ) : \mathcal{H} \otimes \mathcal{H} \to \mathbf{C}$: a non-degenerate even scalar product on $\mathcal{H}$,
(3) $(\ , \dots ,\ )_g : \mathcal{H}^{\otimes n} \to \mathrm{H}^*(\overline{\mathcal{M}}_{g,n}, \mathbf{C})$: correlators for $g, n \geq 0$, $2-2g-n < 0$.

AXIOMS:

(1) S_n-invariance,
(2) the splitting axiom and the genus reduction axiom.

The first axiom means that the correlators are invariant under the evident action of the permutation group S_n acting simultaneously both on $\mathcal{H}^{\otimes n}$ and on $\mathrm{H}^*(\overline{\mathcal{M}}_{g,n}, \mathbf{C})$.

The splitting axiom is the following:

For each g_1, n_1, g_2, n_2 such that $2-2g_i-n_i < 0$ and $n_i > 0$ there is a map

$$\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2} \to \overline{\mathcal{M}}_{g,n}$$

where $g = g_1 + g_2$ and $n = n_1 + n_2 - 2$. Namely, we glue curves $(C; p_1, \dots, p_{n_1})$ and $(C'; p'_1, \dots, p'_{n_2})$ into one singular curve

$$\left(C \bigcup_{p_n = p'_1} C'; p_1, \dots, p_{n_1-1}, p'_2, \dots, p'_{n_2} \right)$$

In this way we identify $\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2}$ with a sub-orbifold in $\overline{\mathcal{M}}_{g,n}$ of complex codimension one (divisor).

The spitting axiom says that

$$(\phi_{i_1} \dots \phi_{i_n})_{g|\overline{\mathcal{M}}_{g_1,n_1} \times \overline{\mathcal{M}}_{g_2,n_2}} =$$

$$= \sum_j \left(\phi_{i_1} \dots \phi_{i_{n_1}} \chi_j\right)_{g_1} \otimes \left(\chi^j \phi_{i_{n_1+1}} \dots \phi_{i_n}\right)_{g_2}$$

where $\{\chi_j\}$ and $\{\chi^j\}$ are dual bases of $\mathcal{H}$ with respect to the scalar product $(\ ,\ )$.

We demand also an analogous property for the restriction of cohomology classes on $\overline{\mathcal{M}}_{g,n}$ to $\overline{\mathcal{M}}_{g-1,n+2}$ via the map

$$(C; p_1, \dots, p_{n+2}) \mapsto (C/(p_1 = p_2); p_3, \dots, p_{n+2})$$

glueing first two marked points into one singular point.

In fact, all this is only a first approximation to the complete story, because one wants to consider so called gravitational descendants $\phi_{i,d}$ where $\phi_i \in \mathcal{H}$ are fields and $d \geq 0$ is an integer. Naively one can define correlators between gravitational descendants as

$$\int_{\overline{\mathcal{M}}_{g,n}} (\phi_{i_1} \dots \phi_{i_n})_g \prod_{i=1}^{n} \left(c_1(T_{p_i} C)\right)^{d_i}$$

In reality, gravitational descendants are more complicated objects, but one can prove using certain recursion relations proven/postulated by physicists that all correlators between gravitational descendendants can be computed using only our generalized correlators with values in cohomology groups. The string partition function (after E. Witten, [W2]) is the generating function for these numbers and it depends on infinitely many variables. E. Witten conjectured in a bit vague form that this function is equal to some τ-function for an integrable hierarhy.

Gromov-Witten invariants.

Let V be a closed symplectic manifold with sufficiently large cohomology class $[\omega] \in \mathrm{H}^2(V, \mathbf{R})$ of the symplectic form ω. The exact meaning of words "sufficiently large" will become clear soon.

One expects after E. Witten ([W1,W2], based on ideas of M. Gromov, [G]) that there is an associated topological σ-model. The space of observables in this theory is the sum of cohomology groups of V:

$$\mathcal{H} := \bigoplus_k \mathrm{H}^k(V; \mathbf{C})$$

with the natural $\mathbf{Z}_2$-grading. The scalar product on $\mathcal{H}$ is the usual Poincaré pairing:

$$(\phi, \psi) := \int_V \phi \wedge \psi$$

Correlators are

$$(\phi_{i_1} \dots \phi_{i_n})_g :=$$

$$= \sum_{\beta \in \mathrm{H}_2(V;\mathbf{Z})} \exp\left(-\int_\beta \omega\right) \times \int_{V^n} (\phi_{i_1} \otimes \dots \otimes \phi_{i_n}) \wedge \mathrm{P.d.}\,(I_{g,h,\beta})$$

Here *P.d.* denotes the Poinaré duality isomorphism between homology and cohomology groups of oriented manifolds and

$$I_{g,h,\beta} \in \mathrm{H}_D\left(\overline{\mathcal{M}}_{g,n} \times V^n; \mathbf{Q}\right) \ , \quad D := (1-g)(\mathrm{dim}V - 6) + 2n + 2\int_\beta c_1(T_V)$$

is called the *Gromov-Witten* invariant of symplectic manifold V.

Geometric meaning of Gromov-Witten invariants.

To define $I_{g,h,\beta}$ let us pick an almost-complex structure $J : TV \longrightarrow TV, \ \ J^2 = -1$ on V compatible with the symplectic structure in the obvious way:

$$\omega(v, Jv) > 0 \text{ for } v \neq 0, \ \ \omega(u,v) = \omega(Ju, Jv), \quad u, v, \in T_x V \ \ .$$

Then consider the space $Y_{g,n,\beta}$ of equivalence classes of holomorphic maps

$$\phi : C \longrightarrow V$$

from complex curves C of genus g with n marked points $p_1, \dots, p_n$ to V representing homology class $\beta \in \mathrm{H}_2(V; \mathbf{Z})$:

$$\phi_*([C]) = \beta$$

The space $Y_{g,n,\beta}$ maps to $\mathcal{M}_{g,n} \times V^n \subset \overline{\mathcal{M}}_{g,n} \times V^n$:

$$(\phi; C; p_1, \dots, p_n) \mapsto (C; p_1, \dots, p_n) \times (\phi(p_1), \dots, \phi(p_n))$$

It is easy to see using the Index Theorem that the dimension of $Y_{g,n,\beta}$ at eadch point is greater than or equal to D. Let us assume that the dimension of $Y_{g,n,\beta}$ is exactly D. Then this space is naturally oriented.

Then we compactify space $Y_{g,n,\beta}$ and define $I_{g,h,\beta}$ as the image of its fundamental class. There are several technical problems here: what to do if the dimension of the space of maps is greater than the expected one, how to prove the independence of invariants on the choice of J and how to check axioms of topological σ-model?

At the moment there are two different approachs to these problems.

The first solution by Y. Ruan and G. Tian (see [RT]) was formulated in somewhat different terms. The main idea is to perturb generically Cauchy-Riemann equations and use Gromov's compactness theorems. This approach works only in the case of manifolds with $c_1 \geq 0$ (in the sense that $\int_C c_1(T_V) \geq 0$ for holomorphic curves C in V) and essentially only in the case of genus equal to zero. Also, with this approach it is very hard to compute Gromov-Witten invariants effectively for complex projective algebraic varieties.

The second approach (M.K., see [K2]) was designed to work for all genera and for all manifolds. The main idea was to introduce a new compactification of the space of holomorphic maps.

Definition. *Stable map $(\phi; C; p_1, \dots, p_n)$ is a holomorphic map ϕ from a marked connected compact curve C to V such that*

(1) *singularities of C are ordinary double points,*
(2) *marked points p_i are pairwise distinct non-singular points of C,*
(3) *the group of automorphisms of $(C; p_1, \dots, p_n)$ commuting with ϕ is finite.*

The last condition is equivalent to the statement that each contracted (= mapped to a point of V) component of C is stable in the Deligne-Mumford sense.

We define $\overline{\mathcal{M}}_{g,n}(V;\beta)$ as the moduli space of stable maps from curves with arithmetic genus g and degree β. We consider it not just as a topological space but as a stack (in a sense, we don't want to forget about finite automorphism groups). Notice that $\overline{\mathcal{M}}_{g,n}(V;\beta)$ is defined in the case $2-2g-n \geq 0$ if $\beta \neq 0$.

Basic properties of $\overline{\mathcal{M}}_{g,n}(V;\beta)$ are collected in the following

Theorem.

(1) *for compact V the stack $\overline{\mathcal{M}}_{g,n}(V;\beta)$ is compact,*
(2) *at a vicinity of each point the space $\overline{\mathcal{M}}_{g,n}(V;\beta)$ can be represented as $X_1 \bigcap X_2$ where X_1, X_2 are two sub-orbifolds in an auxiliary smooth orbifold Y such that $\dim X_1 + \dim X_2 - \dim Y = D$,*
(3) *if V is homogeneous (for example,projective space or a flag variety) then $\overline{\mathcal{M}}_{0,n}(V,\beta)$ is smooth stack (= it is an orbifold).*

First two properties after some work produce certain "virtual fundamental class" which is an element of $\mathrm{H}_D(\overline{\mathcal{M}}_{g,n}(V;\beta), \mathbf{Q})$. The third property means that in the case of flag varieties and genus zero one can use naive definitions of numbers of curves.

Recently we checked all desired properties for the second approach. The only drawback is the lack of the control on the integrality of GW-classes.

The general problem with Gromov-Witten invariants is the convergence question. Why the series

$$\sum_{\beta \in \mathrm{H}_2(V;\mathbf{Z})} \exp(-\int_\beta \omega) \ldots$$

converges?

In all examples it is the case if one multiplies $\beta \in \mathrm{H}^2(V, \mathbf{R})$ by a sufficiently large positive real number. In general, one expects that the σ-model is defined in an open domain in the space of all symplectic structures. There is a way to avoid the problem of convergence: to work over appropriate algebras of formal power series instead of $\mathbf{C}$ (so called Novikov's rings).

Associativity equation.

Genus zero part of Gromov-Witten invariants can be encoded into an analytic function (prepotential, or the genus zero partition function) on the Hilbert space $\mathcal{H}$ considered as a complex supermanifold:

$$F_0(\gamma) := \sum_{n=3}^{\infty} \frac{1}{n!} \int\limits_{\overline{\mathcal{M}}_{g,n}} (\gamma, \ldots, \gamma)_0$$

In all examples this series converges in a non-empty open domain $\mathcal{U}$ of $\mathcal{H}$. The third derivative of F_0 at each point γ of $\mathcal{U}$ can be considered as a symmetric 3-tensor:

$$\partial^3 F_{0|\gamma} \in S^3\mathcal{H}^* = S^3 T^* \mathcal{U}_{|\gamma}$$

The scalar product $(\ ,\ )$ on $\mathcal{H}$ can be used to raise one index. Thus we can consider $\partial^3 F_{0|\gamma}$ as a bilinear operation on the tangent space to $\mathcal{U}$. When restricted to points γ in $\mathrm{H}^2(V, \mathbf{C}) \subset \mathcal{H}$ this operation is called usually the quantum product on $\mathcal{H}$.

The technique of stable maps gives a modification of the prepotential by a polynomial of degree 2. This modificataion has the same third derivative but in examples it gives shorter formulas. Namely, we redefine F_0 as

$$F_0(\gamma) := \sum_{\beta \in \mathrm{H}_2(V;\mathbf{Z})} \exp\left(-\int_\beta \omega\right) \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\overline{\mathcal{M}}_{g,n}(V;\beta)} \prod_{i=1}^{n} p_i^*(\gamma) \ .$$

The prepotential satisfies so called Witten-Dijkgraaf-Verlinde-Verlinde equation (see [DVV] and [W2]). This very remarkable equation means that the product on $T\mathcal{U}$ is associative. The quantum product is automatically commutative and has a unit equal to $1 \in \mathbf{C} = \mathrm{H}^0(V, \mathbf{C}) \subset \mathcal{H}$.

In our formalism the associativity follows immediately from the splitting axiom and from certain linear relation among components of the compactification divisor of $\overline{\mathcal{M}}_{0,n}$. Denote by D_S for $S \subset \{1, \dots, n\}$, $2 \le \#S \le n-2$, the divisor in $\overline{\mathcal{M}}_{0,n}$ which is the closure of the moduli of stable curves $(C; p_1, \dots, p_n)$ consisting of two irreducible components C_1, C_2 such that $p_i \in C_1$ for $i \in S$ and $p_i \in C_2$ for $i \notin S$.

Lemma. *We have the following identity in $H^2(\overline{\mathcal{M}}_{0,n}, \mathbf{Z})$*

$$\sum_{\substack{S:1,2\in S\\ 3,4\notin S}} [D_S] = \sum_{\substack{S:1,3\in S\\ 2,4\notin S}} [D_S] \ .$$

Both sides in the equality above are pullbacks under the forgetful map $\overline{\mathcal{M}}_{0,n} \to \overline{\mathcal{M}}_{0,4}$ of points $D_{\{1,2\}}$, $D_{\{1,3\}} \in \overline{\mathcal{M}}_{0,4} \simeq \mathbf{P}^1$. It is clear that any two points on $\mathbf{P}^1$ are rationally equivalent as divisors. $\square$

Every cohomology class of $\overline{\mathcal{M}}_{0,n}$ can be obtained as an intersection of several divisors D_S. It follows that whole genus zero part of correlators is determined by top degree classes, i.e. the Taylor coefficients of F_0. In fact, the associativity equation for F_0 is equivalent to the possibility to reconstruct all lower degree classes:

Theorem (M.K.+Yu. Manin, R. Dijkgraaf+E. Getzler). *Genus zero part of the axioms of the topological sigma-model is equivalent to the associativity equation on the formal power series F_0.*

This theorem was proved via a complicated induction using results of S. Keel (see the reference in [KM]) about cohomology algebras of spaces $\overline{\mathcal{M}}_{0,n}$. Recently E. Getzler obtained another proof using Hodge theory. There are two immediate corollaries of this theorem:

(1) coupling with gravitational descendants in genus zero is completely determined by the series F_0,
(2) there exists an algebraic procedure which made from two solutions of WDVV equations F_0 and F_0' in vector spaces $\mathcal{H}$ and $\mathcal{H}'$ respectively a new solution F_0"$\times$"F_0' in the vector space $\mathcal{H} \otimes \mathcal{H}'$.

Cohomology-valued correlators in the tensor product of two toplogical field theories are defined as products of correlators in the cohomology rings. The operation of the tensor product on the level of functions F_0 can not be completely elementary. For example, the prepotential for $V = \mathbf{C}P^1$ is very simple:

$$F_0(t_0, t_1) = \frac{t_0^2 t_1}{2} + e^{t_1}$$

Nevertheless, the prepotential for $V \times V$ is quite non-trivial (see [DFI]).

Applications of the associativity equation.

The simplest interesting case is $V = \mathbf{C}P^2$. The Hilbert space of the topological theory is 3-dimensional:

$$\mathcal{H} = \mathbf{C}^3 = \mathrm{H}^0(V, \mathbf{C}) \oplus \mathrm{H}^2(V, \mathbf{C}) \oplus \mathrm{H}^4(V, \mathbf{C})$$

We introduce three coordinates t_0, t_1, t_2 in $\mathcal{H}$ according to the decomposition above. The prepotential has the form

$$F_0(t_0, t_1, t_2) = \frac{t_0^2 t_2 + t_0 t_1^2}{2} + \sum_{d=1}^{\infty} n_d \frac{t_2^{3d-1}}{(3d-1)!} e^{dt_1}$$

Here the first summand comes from the contribution of the constant maps and coefficients n_d denote the number of rational curves of degree d passing through generic $3d-1$ points on $\mathbf{C}P^2$. After the substitution of this general formula into the associativity equation we obtain the following relation among numbers n_d:

$$\text{for } d \geq 2 \quad n_d = \sum_{a,b \geq 1:\, a+b=d} n_a n_b \left(\frac{(3d-4)!a^2b^2}{(3a-2)!(3b-2)!} - \frac{(3d-4)!a^3 b}{(3a-1)!(3b-3)!} \right)$$

Thus, it is enough to know only one number n_1 which is the number of lines passing through two points, i.e. $n_1 = 1$. By recursion we obtain immediately

$$n_2 = 1,\ n_3 = 12,\ n_4 = 620,\ n_5 = 87304,\ n_6 = 26312976,\ \dots$$

First three values are classical in algebraic geometry. Number $n_4 = 620$ was computed in 1873 by H. Zeuthen. Numbers $n_{\geq 5}$ are all new.

In general, one can associate with a solution of the WDVV equation associated with a symplectic manifold an isomonodromy deformation of a linear system of differential equations with regular singularities on $\mathbf{C}P^1$ in at most $\dim \mathcal{H} + 1$ points. In all computed examples the monodromy of this auxiliary equation takes values in the group of integer matrices $GL(N, \mathbf{Z})$. There is a physical explanation for it proposed by S. Cecotti and C. Vafa (see [CV]) based on general properties of massive $N = 2$ models. Mathematicians expect that any linear differential euation with an integral monodromy and algebraic coefficients is equivalent to a Picard-Fuchs equation. This means that one can expect a kind of mirror symmetry for generic symplectic manifolds, not necessarily Calabi-Yau ones.

In the case $V = \mathbf{C}P^2$ this isomonodromy deformation can be transformed into the Painlevé VI equation. Using this fact one can find the asymptotic of numbers n_d:

$$\frac{n_d}{(3d-1)!} \sim \ const_1 \cdot d^{-7/2} e^{const_2 d}(1 + o(1))$$

The reason for this is a kind of quasi-homogeneity of prepotential following from the Riemann-Roch formula (= the Index theorem) for dimensions of spaces of maps. In physics it correponds to the renormalization group flow. Let us introduce homogeneous coordinates in $\mathbf{Z}$-graded vector space $\mathcal{H}$. We define vector field L_0 on $\mathcal{U}$ by formula

$$L_0 := \sum_{deg\ t_i \neq 2} (deg\ t_i - 2) \frac{t_i \partial}{\partial t_i} - \sum_{deg\ t_i = 2} 2(Coefficient_i\ c_1(T_V)) \frac{\partial}{\partial t_i}$$

Then we have

$$L_0(F_0) = (dim_{\mathbf{R}} V - 6)F_0 + \ quadratic \ \ terms$$

Using the quantum product on the tangent space to $\mathcal{H}$ one can construct powers of the vector field L_0:

$$L_n := L_0 \times L_0 \times \cdots \times L_0 \quad (n+1 \quad times)$$

for $n \geq -1$. The remarkable fact is that vector fields L_n on $\mathcal{H}$ form an action of the Witt algebra (= the Lie algebra of polynomial vector fields on the line):

$$[L_n, L_m] = (m-n)L_{n+m}$$

Theorem (B. Dubrovin, [D1,D2]). *If L_0 is diagonalizable in an open set then the associativity equation is a completely integrable system. Solutions of this system depend on a finite number of parameters.*

This theorem is applicable to many varieties with $c_1 > 0$, including projective spaces, Grassmanians etc.

Example from the Mirror symmetry.

Let V be a quintic 3-dimensional hypersurface in $\mathbf{C}P^4$. It is a Calabi-Yau manifold, i.e. a complex manifold with $c_1 = 0$. The prepotential F_0 is essentially encoded in a function of one variable:

$$F(t) := \frac{5}{6}t^3 + \sum_{d=1}^{\infty} N_d^{phys} \exp{(dt)}$$

Here again the first coefficient comes from constant maps and other coefficients are

$$N_d^{phys} := N_d + \frac{1}{2^3}N_{d/2} + \frac{1}{3^3}N_{d/3} + \ldots \ \in \mathbf{Q}$$

Numbers $N_d \in \mathbf{Z}$ are (conjecturally) numbers of rational curves of degree d on V. The formula for N_d^{phys} arised from the consideration of multiple ramified coverings of rational curves in V as maps from $\mathbf{C}P^1$ to V. The associativity equation gives no restriction on the sequence N_d because all rational curves are isolated and do not intersect each other.

The famous mirror formula of P. Candelas et al. (see [COGP] and [Y]) can be described as follows. Let following four functions

$$\psi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n$$

$$\psi_1(z) = \ \log\, z \cdot \psi_0(z) + 5 \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \left(\sum_{k=n+1}^{5n} \frac{1}{k} \right) z^n$$

$$\psi_2(z), \psi_3(z) = \ldots$$

form a basis of the space of solutions of the differential equation

$$\left(\left(z\frac{d}{dz}\right)^4 - 5z(5z\frac{d}{dz}+1)(5z\frac{d}{dz}+2)(5z\frac{d}{dz}+3)(5z\frac{d}{dz}+4)\right)\psi(z) = 0 \ .$$

Then

$$F\left(\frac{\psi_1}{\psi_0}\right) = \frac{\psi_1\psi_2 + \psi_0\psi_3}{\psi_0^2} \ .$$

The mirror conjecture for quintics for a long time was not formulated well mathematically because it is still unknown whether the number of rational curves in each degree is finite (conjecture of H. Clemens) and how to count contributions of singular curves.

Calculations show, at least for first few degrees, that the definition with stable maps give automatically the "physical number" of rational curves. We used the following approach to the computational definition of N_d^{phys}.

Denote by X_d the space of stable maps of degree d of genus zero curves without marked points to $\mathbf{C}P^4$:

$$X_d := \overline{\mathcal{M}}_{0,0}(\mathbf{C}P^4, d[\mathbf{C}P^1])$$

It is a compact complex orbifold of complex dimension $5d+1$. Let us introduce a vector bundle $\mathcal{E}_d$ of rank $5d+1$ on X_d (in the orbifold sense too). The fiber of $\mathcal{E}_d$ at the point of X_d representing a map $\phi : C \to \mathbf{C}P^4$ is defined as the space of global sections $\Gamma(C, \phi^*\mathcal{O}(5))$.

Our definition of the number of curves is

$$N_d^{phys} := \int_{X_d} c_{5d+1}(\mathcal{E}_d)$$

The explanation of this formula is the following. The quintic V is given by an equation of degree 5:

$$Q(x_1 : x_2 : x_3 : x_4 : x_5) = 0$$

We can consider polynomial Q as a global section of the line bundle $\mathcal{O}(5)$ on $\mathbf{C}P^4$. It induces in the evident way a global section Q_d of the vector bundle $\mathcal{E}_d$. Zeroes of Q_d coincide with the moduli of stable maps to V. If we perturb generically the section Q_d it will have finitely many zeroes. The number of these zeroes counted with signs coming from the natural orientation is equal to the integral of the Euler class of the bundle $\mathcal{E}_d$.

We calculated using computer first four values of N_d^{phys}, and these numbers agreed with the mirror formula. Moreover, we obtained a closed formula for the function $F(t)$ thus reducing the Mirror conjecture to an explicit identity. We will describe the general scheme of computations in the next section.

Toric methods.

Almost all examples of Calabi-Yau manifolds constructed up to now are obtained by a resolution of singularities of complete intersections in toric varieties. V. Batyrev and L. Borisov in [BB] proposed a general form of the mirror symmetry for this class of manifolds. The combinatorial counterpart of the mirror symmetry in the toric picture is the usual duality of convex cones in real vector spaces (a generalization of the Legendre transform).

We will give here a sketch of a procedure giving a closed formula for Gromov-Witten invariants of arbitrary complete intersections in toric varieties, not only Calabi-Yau ones. We will illusrtate our method in the case of quintics in $\mathbf{C}P^4$.

The complex algebraic torus $\mathbf{T} := (\mathbf{C}^\times)^5$ acts diagonally on $\mathbf{C}^5$, hence on $\mathbf{C}P^4$, on the line bundle $\mathcal{O}(5)$, on variety X_d and on the vector bundle $\mathcal{E}_d$. In $\mathbf{T}$-equivariant case one can use a classical formula of R. Bott (see [B]) reducing the computation of the integral of any characteristic class of the bundle to the integral over the submanifold(s) of fixed points:

$$\int_{X_d} Euler(\mathcal{E}_d) = \sum_{\substack{\text{components} \\ \text{of } (X_d)^{\mathbf{T}}}} \int_{X_d^\alpha} \frac{Euler^{equiv}(\mathcal{E}_d)}{Euler^{equiv}(\mathcal{N}_{X_d^\alpha})}$$

Here $Euler^{equiv}$ denotes the equivariant Euler class of a bundle, index α numerates connected components of the set $(X_d)^{\mathbf{T}}$ of fixed points and $\mathcal{N}_{X_d^\alpha}$ denotes the nornal bundle to the corresponding component. The right hand side of this identity takes values apriori in the field of rational functions on the vector space $\mathbf{C}^5$ which is the Lie algebra of $\mathbf{T}$. In fact, the r.h.s. is a constant.

Each point of $(X_d)^{\mathbf{T}}$ is representing a map ϕ from a tree of projective lines $(\mathbf{C}P^1)_k$ to $\mathbf{C}P^4$ with the image of each irreducible component $(\mathbf{C}P^1)_k$ invariant under the action of $\mathbf{T}$. There are two types of components:

(1) contracted ones, $\phi((\mathbf{C}P^1)_k)$ is one of 5 fixed points $\{x_1, \ldots, x_5\} = (\mathbf{C}P^4)^{\mathbf{T}}$,
(2) non-contracted ones, $\phi((\mathbf{C}P^1)_k)$ is one of 10 lines passing through two fixed points and the map ϕ in some homogeneous $\mathbf{T}$-invariant coordinates looks like
$$(z_1 : z_2) \mapsto (z_1^d : z_2^d : 0 : 0 : 0)$$

We associate a new tree Γ with such a map by declaring connected components of $\phi^{-1}(\{x_1, \ldots, x_5\})$ to be the vertices and non-contracted components to be the edges. The vertices of Γ are labeled by indices $1, \ldots, 5$ and the edges are labeled by degrees $d = 1, 2, \ldots$.

One can see that the connected components of $(X_d)^{\mathbf{T}}$ are in one-to-one correspondence with the labeled trees. After a long calculation (using intersection theory on $\overline{\mathcal{M}}_{0,n}$) one can find the contribution of each component into the Bott formula. This contribution is a product of local weights. Thus, the whole generating function is equal to the sum over tree diagrams for some abstract "Lagrangian". The total sum is equal to the critical value of the action functional. Still the formula is pretty complicated. Using some tricks we reduced the problem to the compuation of the critical value of a new functional which is quadratic in all variables except of a finite number of them.

Now the problem of counting of rational curves on quintics is reduced essentially to the inversion of an integral operator with the kernel expressed via a generalized hypergeometric function. We reproduce here from [K2] the resulting functional:

$$S = S(t, \lambda_i; \phi_{ij,d}, \mu_i) =$$

$$\frac{1}{2} \sum_{\substack{i,j;d \\ i \neq j}} \frac{d^3(\lambda_i - \lambda_j)^2 \prod_{a+b=d:a,b\geq 1} \prod_{k=1}^{5} (a\lambda_i + b\lambda_j - d\lambda_k)}{\prod_{a+b=5d:a,b\geq 1} (a\lambda_i + b\lambda_j)} exp\left(-td - \frac{\xi_i - \xi_j}{\lambda_i - \lambda_j}\right) \phi_{ij,d}\phi_{ji,d} +$$

$$+\frac{1}{2}\sum_{\substack{i,j,d,j',d'\\ j,j'\neq i}} \nu_i \frac{\phi_{ij,d}\phi_{ij',d'}}{\frac{d}{\lambda_i-\lambda_j}+\frac{d'}{\lambda_i-\lambda_{j'}}} -$$

$$-\sum_{\substack{i,j,d\\ i\neq j}} \nu_i \frac{\lambda_i-\lambda_j}{d}\xi_i\phi_{ij,d} + \sum_{\substack{i,j,d\\ i\neq j}} \nu_i \frac{(\lambda_i-\lambda_j)^2}{d^2}\phi_{ij,d} + \sum_{i=1}^{5} \nu_i \frac{\xi_i^3}{6}\ .$$

Here indices i, j ran from 1 to 5 and d is a positive integer. Variables ν_i are defined as

$$\nu_i := \frac{5\lambda_i}{\prod_{j:j\neq i}(\lambda_i-\lambda_j)}$$

The critical value of S as a function in variables $\phi_{ij,d}, \mu_i$ is independent of λ_i and equal to $F(t) - 5t^3/6$, the generating functions for numbers of rational curves on quintics.

Applications of the Bott formula.

(1) The heuristic formula of P. Aspinwall and D. Morrison (see [AM]) for the contribution of multiple coverings of rational curves in 3-dimensional Calabi-Yau varieties was checked by Yu. Manin in [M] using summation over trees.
(2) One can get a formula for the genus-zero partition function for the complete intersections in toric varieties (Batyrev-Borisov's series of mirrors).
(3) One can count (in principle) numbers of higher genus curves in $\mathbf{C}P^n$, and, more generally, in toric varieties.

The problem in the second and the third applications is that for non-homogeneous toric variety V or if $g \geq 1$ spaces $\overline{\mathcal{M}}_{g,n}(V;\beta)$ are singular. Still one can try to apply the Bott formula because subspaces of fixed points $\overline{\mathcal{M}}_{g,n}(V;\beta)^{\mathbf{T}}$ are always smooth. In the case of projective space and genus zero curves the class in the equivariant K-theory of the normal bundle to $\overline{\mathcal{M}}_{g,n}(V;\beta)^{\mathbf{T}}$ can be decomposed into the sum of certain standard piece and of the bundle with the fiber equal to $\mathrm{H}^0(C, \phi^* T_V)$. In general situation it is not a vector bundle. Nevertheless, the formal difference

$$\mathrm{H}^0(C, \phi^* T_V) - \mathrm{H}^1(C, \phi^* T_V)$$

in the equivariant K-theory is well-defined and we can use it formally. This procedure gives again some explicit formulas. We hope that in a future it will be possible to obtain KP-hierarchy from representations of generating functions via infinite-dimensional determinants.

For the calculations with higher genus curves in the "universal" toric formula one has to compute certain integral over the moduli spaces of stable curves:

$$F(x_*, y_*, z_*, \lambda) =$$

$$= \sum_{\substack{g,n\\ 2-2g-n<0}} \frac{\lambda^{2g-2}}{n!} \sum_{\substack{\alpha_1,\dots,\alpha_g\\ \beta_1,\dots,\beta_n}} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\prod_{j=1}^{g}(x_{\alpha_j}+h_j)}{\prod_{i=1}^{n}(y_{\beta_i}+z_{\beta_i}c_1(T_{p_i}C))}$$

where formal variables h_j are defined via equality

$$\prod_{j=1}^{g}(1+h_j)=\sum_{k=0}^{g}c_k(\mathcal{H}_g)\ \in \mathrm{H}^*(\overline{\mathcal{M}}_{g,n},\mathbf{Q})$$

Here $\mathcal{H}_g$ is the g-dimensional vector bundle over $\overline{\mathcal{M}}_{g,n}$ with the fiber over C equal to $\mathrm{H}^1(C,\mathcal{O}_C)$. One can hope in analogy with the pure gravity case (Witten's conjecture, see [W2] and [K1]) that this universal function F is equal to the free energy of some matrix model.

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